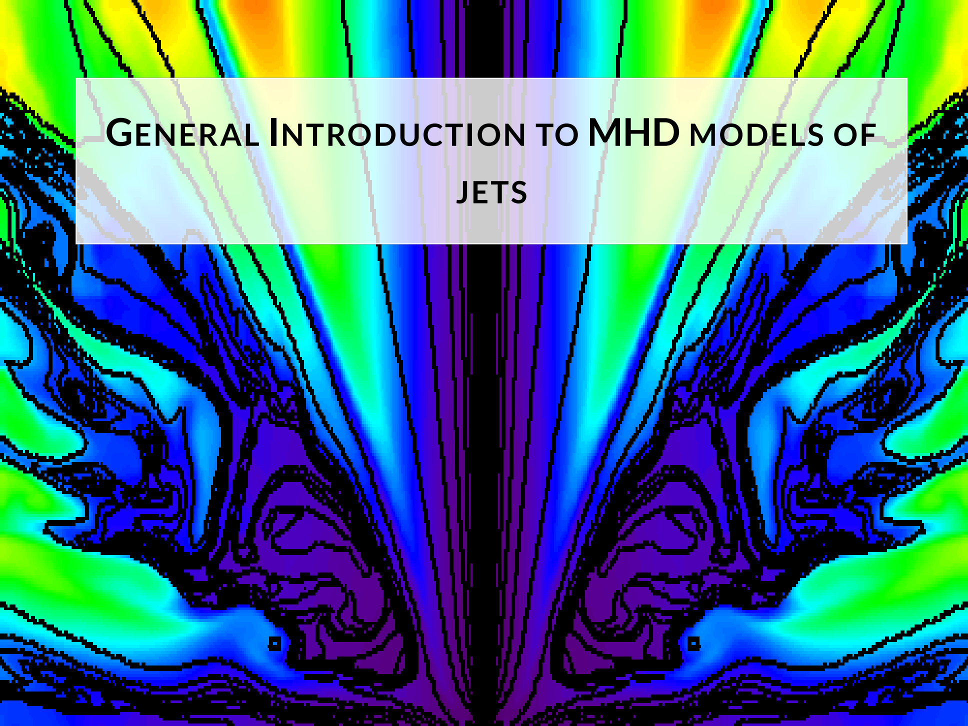
General Introduction to MHD models of jets



What is MHD? Why do we need MHD?

MAGNETO-HYDRO-DYNAMICS

MAGNETO $\Longrightarrow$ magnetic field involved

HYDRO $\Longrightarrow$ electrically conductive fluid, e.g. plasmas

DYNAMICS $\Longrightarrow$ study of the forces and torques determining the motion

With the exception of planet's atmosphere, plasmas and magnetic fields are common ingredients of each and every astrophysical system!

MHD and Jets

Where do we find plasma and magnetic fields in astrophysical systems?

- Accretion
- Wind
- Jets

Each of these phenomena can be described by a system of MHD non-linear equations.

MHD approximation

MHD approximation $=$ simplified fluid mechanics $+$ Maxwell's equations

ASSUMPTIONS:

- Fluid approximation: local thermodynamic quantities can be meaningfully defined in the plasma, and variations in these quantities are slow compared with the timescale of the microscopic processes in the plasma.
- In the plasma there is a local, instantaneous relation between electric field and current density (Ohm's law).
- The plasma is electrically neutral.

GOAL:
Solution of equilibrium of forces perpendicular (Grad-Shafranov eq/TORAMAK) and parallel (Bernoulli eq for polytropic EoS) to the magnetic surfaces.

MAIN ISSUE:
3 critical surfaces, from which constants of motion (CoM) are derived.

Ideal MHD

Ideal MHD: perfectly conductive fluid, i.e. infinite magnetic Reynolds number ($R_m \sim \sigma_0$), therefore E-field$= 0$, but only in the fluid/comoving frame (K')!

MAXWELL EQNS:

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$$

$$\nabla \cdot \mathbf{E} = 4\pi\sigma$$

$$\nabla \cdot \mathbf{B} = 0$$

$$4\pi\mathbf{j} + \partial\mathbf{E}/\partial t = c\nabla \times \mathbf{B} \quad \Longrightarrow \quad \partial\sigma/\partial t + \nabla \cdot \mathbf{j} = 0 \quad \overset{NR}{\Longrightarrow} \quad \nabla \cdot \mathbf{j} = 0$$

$$\partial\mathbf{B}/\partial t = c\nabla \times \mathbf{E} \quad \Longrightarrow \quad \partial\mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B})$$

FLUID DYNAMICS EQNS:

$$F_L = \frac{1}{c}\mathbf{j} \times \mathbf{B} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\varrho\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \varrho\nabla\phi$$

$$\frac{\partial\varrho}{\partial t} + \varrho\nabla \cdot \mathbf{v} = 0$$

$$P = Q\varrho^\gamma$$

Bernoulli and Grad-Shafranov equations

EoM$_p$: Integrating the projection, we get the energy or **Bernoulli equation**

$$\frac{1}{2}v_{p,A}^2 = \begin{pmatrix} kinetic \\ flux \end{pmatrix} + \begin{pmatrix} entalphy \\ flux \end{pmatrix} + \begin{pmatrix} gravtnl \\ flux \end{pmatrix} + \begin{pmatrix} Poynting \\ flux \end{pmatrix}$$

EoM$_\perp(\perp S_B)$: Grad-Schlüther-Shafranov or **Transfield equation**

$$quasi - linear\, PDE\, for\, \psi(\varpi, z)$$

CoMs : Field line constants: $\Omega, \Psi_A, L, \mu c^2, Q$

❢ Highest order derivative terms vanish at AP in GSS eqn!
The **Alfvén regularity condition**: solved for the slope of the solution of the Bernoulli equation at AP, p_A.

❢❢ In the Bernoulli eqn, two more critical points: when $v_{p,A} = v_s$ (**slow**) and $v_{p,A} = v_f$ (**fast**).

ANY REGULAR SOLUTION OF BERNOULLI EQN MUST PASS BOTH MSP, MFP!

Approach: I (Weber & Davis, 1967)

MAIN ASSUMPTIONS: fixed shape for magnetic field, i.e. non GS eq.!

SYSTEM:
- only $\perp$ forces, i.e. Bernoulli equation
- from slow and fast magnetosonic critical surfaces we get 2 CoM:
  - CoM 1: mass-to-magnetic-flux ratio, Ψ_A
  - CoM 2: total energy $E(\psi)$

PROS: determination of asymptotic speeds

CONS: no info on collimation

Approach: II (Suess & Nerney, 1973)

MAIN ASSUMPTIONS: none, but only numerical. Perturbation of a spherically symmetric, iterative methods, etc.

SYSTEM: full system

PROS: true shape of the field lines

CONS: only numerical

Approach: III (Blandford & Payne (1982), Vlahakis & Konigl (2000,2003), etc.)

MAIN ASSUMPTIONS: specific dependence of the flow variables on the independent variable (**self-similarity** assumption)

SYSTEM: full system, but reduced number of independent variables, in most of the cases to just one.

PROS: accounts for the force balance

CONS:

- not regular/valid in the whole parameter space
- not properly accounting for MFP (?)
- singularity for electrical current along the axis of symmetry

Other approaches

IV: Variational approach (Rosso & Pelletier, 1994)

V: Slender jet approximation (Koupelis & van Horn (1989), Koupelis (1990))

... ...

...and this is only until 1999 (Lery at al, 1999)!

Magnetic force and curvature force

Lorentz force is perpendicular to the B-field. Along magnetic field lines, only hydrodynamic forces act.

$$
\begin{aligned}
F_L &= \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
&= -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}
\end{aligned}
$$

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The **curvature force** term for an axisymmetric azimuthally directed field ($=B\hat{\phi}$) in cylindrical coordinates (ϖ, ϕ, z), considering that $\hat{\phi}\cdot\nabla\hat{\phi} = -\hat{\boldsymbol{\varpi}}/\varpi$, looks like

$$\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} = -\frac{B^2}{4\pi}\frac{\hat{\boldsymbol{\varpi}}}{\varpi}$$

The curvature is directed toward the centre of curvature: **HOOP STRESS**!

Stream function

In cylindrical coordinates, an axisymmetric field is constant wrt the azimuthal coordinate, i.e. $\partial \mathbf{B}/\partial\phi = 0$ and we can decompose the field as

$$\mathbf{B} = \mathbf{B}_p(\varpi, z) + \mathbf{B}_t(\varpi, z)$$

in particular

$$\mathbf{B}_p = (B_\varpi, 0, B_z) = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\hat{\boldsymbol{\varpi}} + \frac{1}{\varpi}\frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$
$$\mathbf{B}_t = B_\phi\hat{\boldsymbol{\phi}}$$

where the **flux/stream function** ψ is defined as

$$\psi(\varpi, z) = \int_0^\varpi \varpi B_z d\varpi$$

and it equals, apart for a factor 2π the magnetic flux contained in a circle of radius ϖ.

Alfvén & Magnetosonic waves

(2D-)Problem

homogeneous B-field in a uniform fluid initially at rest

+

small perturbations in $\mathbf{B}$, p, ϱ at later times

Alfvén waves

$\mathbf{B} \to \mathbf{B} + \delta\mathbf{B}$,

$\delta p = \delta\varrho = 0$

Ion waves

$p \to p + \delta p$,

$\varrho \to \varrho + \delta\varrho$,

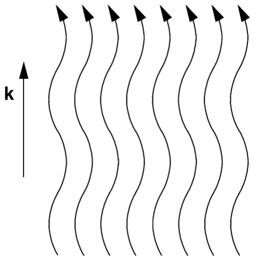
$\delta\mathbf{B} = 0$

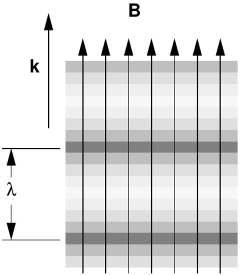
Magnetosonic waves

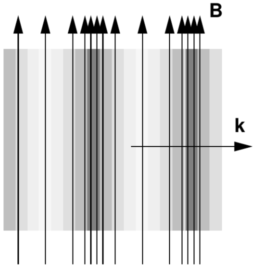
$p \to p + \delta p$,

$\varrho \to \varrho + \delta\varrho$,

$\delta\mathbf{B} = 0$







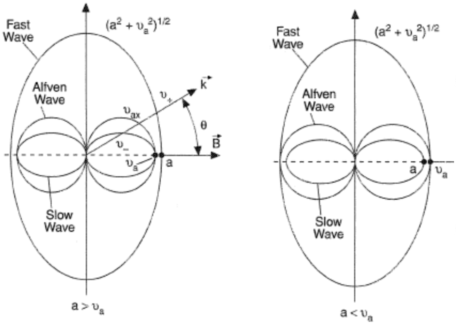
Waves' Properties

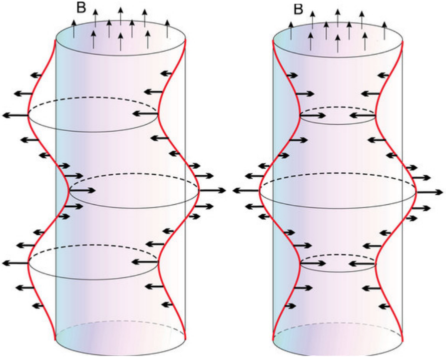
ALFVÉN WAVE

- Incompressional, i.e. $\delta p = 0$
- Torsional/Transversal/Shear, i.e. $\delta \mathbf{B} (\parallel \mathbf{v}) \perp \mathbf{B} (\parallel \mathbf{k})$

SLOW/FAST MAGNETOSONIC WAVE

- Compressional, i.e. $\delta p = 0$
- Longitudinal, i.e. $(\delta \mathbf{B} \parallel \mathbf{v} \parallel \mathbf{k}) \perp \mathbf{B}$



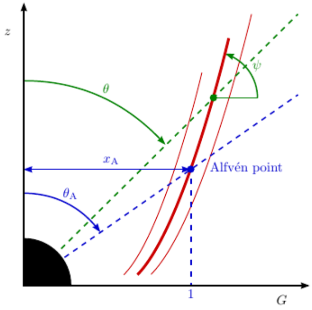


(Radial) Self-similar models

Starting from Blandford & Payne (1982), several self-similar models for accretion/wind/jet have been developed, with common characteristics.

Paper	Gravity	Warm	Relativistic	MSP	AP	MFP
Blandford & Payne (1982)	✓				✓	
Li et al. (1992)			✓		✓	
Vlahakis et al. (2000)	✓	✓		✓	✓	✓
Vlahakis & Königl (2003a)		✓	✓		✓	
Polko et al. (2010)		✓	✓		✓	✓
Polko et al. (2013a)	✓	✓	✓	✓	✓	✓

- Assumptions:
  - ideal, time-independent MHD
  - axisymmetry
  - zero azimuthal E-field
  - no external forces (**self-similarity**)
- Uknown functions:
  - $M(\theta)$
  - $G(\theta)$
  - $\psi(\theta)$
  - $\xi(\theta)$



Determinant Method

By writing the transfield equation and the derivative wrt θ of the energy equation as follows

$$A_1 \frac{dM^2}{d\theta} + B_1 \frac{d\psi}{d\theta} = C_1 \qquad \text{Energy eqn}$$
$$A_2 \frac{dM^2}{d\theta} + B_2 \frac{d\psi}{d\theta} = C_2 \qquad \text{Transfield eqn}$$

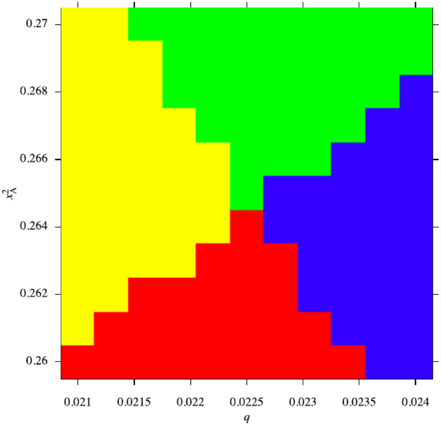
where As, Bs and Cs are functions of θ and all the unknown functions M, ψ, G, ξ. Combining them with the determinant method, we get

$$\frac{dM^2}{d\theta} = \frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - A_2 B_1} \qquad \text{Wind eqn}$$
$$\frac{d\psi}{d\theta} = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1} \qquad \psi \text{ eqn}$$

- If gravity is accounted for, gravity terms are included in C_1 and C_2.

Solution's properties

- Complete regular solutions are crossing smoothly each of the singular point, MSP, AP, MFP.
- Numerator and denominator of the wind equation cross zero in the same point at the same time.
- When a solution crosses one of the point of interest, we can extrapolate information about the system (the velocity of the wave, the radius of the jet/wind, the position of the shock/acceleration/recollimation region, etc.)



Alfvén surface problem

The coefficients As, Bs and Cs at the AS have singularities of the kind $0/0$, that can be eliminated applying de l'Hôpital rule. Unfortunately there are some problems left...

Simply by multiplying the eqns by the factor $f_{Mx} = (1 - M^2 - x^2)$, we can easily demonstrate that the system collapse into a single eqn on AS

$$\not\infty \frac{dM^2}{d\theta} + 0 \frac{d\psi}{d\theta} = \not\infty$$
$$\not\infty \frac{dM^2}{d\theta} + 0 \frac{d\psi}{d\theta} = \not\infty$$

therefore **the determinant method doesn't hold on AS** and the derivative of ψ is not determined.

Isolating the infinities

By noticing that

$$\left(A_{1,\infty} + A_{1,\not\infty}\right)\frac{dM^2}{d\theta} + B_1\frac{d\psi}{d\theta} = \left(C_{1,\infty} + C_{1,\not\infty}\right)$$

$$\left(A_{2,\infty} + A_{2,\not\infty}\right)\frac{dM^2}{d\theta} + B_2\frac{d\psi}{d\theta} = \left(C_{2,\infty} + C_{2,\not\infty}\right)$$

and, that after rearranging terms, we have

$$\left(A_{1,\infty}\frac{dM^2}{d\theta} - C_{1,\infty}\right) + A_{1,\not\infty}\frac{dM^2}{d\theta} + B_1\frac{d\psi}{d\theta} = C_{1,\not\infty}$$

$$\left(A_{2,\infty}\frac{dM^2}{d\theta} - C_{2,\infty}\right) + A_{2,\not\infty}\frac{dM^2}{d\theta} + B_2\frac{d\psi}{d\theta} = C_{2,\not\infty}$$

$$\Downarrow \qquad\qquad \Downarrow$$

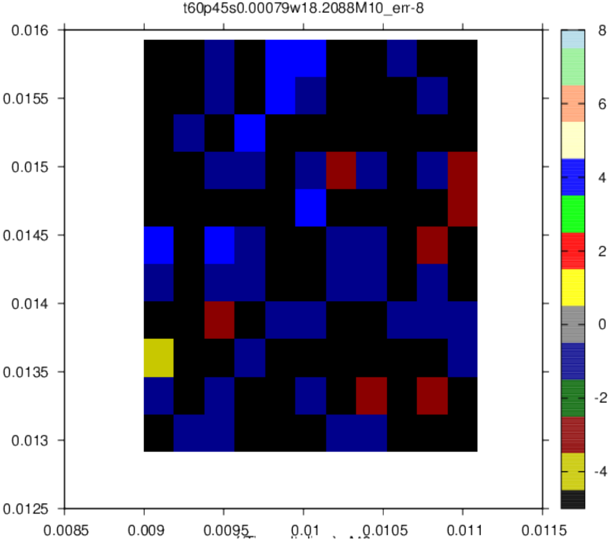
$$\left.\frac{dM^2}{d\theta}\right|_A = \left.\frac{C_{1,\infty}}{A_{1,\infty}}\right|_A = \left.\frac{C_{2,\infty}}{A_{2,\infty}}\right|_A = p_A$$

ARC

$$A_{1,\not\infty}\frac{dM^2}{d\theta} + B_1\frac{d\psi}{d\theta} = C_{1,\not\infty}$$

$$A_{2,\not\infty}\frac{dM^2}{d\theta} + B_2\frac{d\psi}{d\theta} = C_{2,\not\infty}$$

Currents status of the new solutions



Conclusions

- In most of the astrophysical systems we observe evidence of magnetized outflows/inflows of plasma
- We study the phenomenon using the MHD approximation formalism.
- Several approaches can be adopted to solve the MHD system of equation.
- Self-similar model have some disadvantages amongst which the fuzzy parameter space seems to be the most difficult to overcome.
- Using the determinant method, removing by hand the singularity at the AP, gives us a few solutions, all of them in the same narrow region of the parameter space.
- The new formulation of the equations that tries to handle the singularity at the AP and avoid using the determinant method, hopefully will provide a much larger number of solutions by allowing us to explore a wider region of the parameter space.